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Analytical Methods for Performance Evaluation of Nonlinear Filters*

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I. INTRODUCTION

Following the success of the Kalman–Bucy linear filter theory [1], nonlinear filtering received considerable attention in the mathematical and engineering literature in the past few years advancing the state space approach to nonlinear filtering problems. As a main theoretical result, it has been recognized that the truly optimal nonlinear filters are infinite dimensional [2–7]. Thus, any practical (finite dimensional) nonlinear filter algorithm is necessarily suboptimal in the sense of being an approximation to the truly optimal nonlinear filters. Several different schemes were set forth to obtain finite dimensional (suboptimal) realizations for the optimal nonlinear filters [8–17]. However, all hitherto known suboptimal nonlinear filter algorithms have two important factors in common. (i) *Stochastic description* for the approximate statistics of the filtering error; this usually requires extensive Monte Carlo type numerical studies to evaluate the filters' performance. (ii) *Structural complexity*, since the differential (or difference) equations describing the estimate of the state vector and the statistics of the estimation error form a coupled system of equations; in many cases, this poses severe difficulties in implementing the nonlinear filter schemes. (These factors sharply contrast the features of the Kalman–Bucy optimal linear filter algorithm.) An account on the present state of art in modern filtering theory can be found in [24].

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In many cases of application, it is desirable to construct suboptimal nonlinear filter schemes having tractable structural complexity corresponding to the constraints of a particular application and, to evaluate the filters' performance by analytical techniques in parametric forms. The present investigation is addressed to the question of developing analytical methods for evaluating the performance of suboptimal nonlinear filters such that the filters' structure is fixed by postulating a simple form for it.

In the present investigation, the filtering problem is considered in the continuous time domain. The postulated simple suboptimal nonlinear filter structure closely parallels the structure of the Kalman–Bucy optimal linear filter algorithm. Two filter performance evaluation methods are developed based on the Kolmogorov equations for the transition density of Markov processes. The expansions in the approximations for the nonlinear system and observation functions are *in effect* carried out up to second-order terms in both methods. The difference between Method I and Method II is the sequencing of expansions and averaging. The description of the filters' performance is sought in terms of second-order statistics (mean and covariance) in both methods. The equations for the mean and covariance of the filtering error resulting from Method I and Method II are different. The resulting equations of both methods have, however, an important common feature: They are *deterministic* differential equations describing the time evolution of the mean and covariance of the filtering error process for the fixed filter structure in terms of the known (postulated) filter gain and system and noise parameters.

The developed deterministic differential equations can also be utilized to determine appropriate (deterministic) filter gains for the fixed structure nonlinear filter. The salient features of the new performance evaluation (and filter gain construction) methods are illustrated on two examples.

II. Problem Formulation

Let the dynamical model of a continuous $x(t)$ process be given by a set of stochastic differential equations interpreted in the sense of Ito [18, 19]:

$$dx = f(t, x)\, dt + D(t)\, d\xi, \tag{1}$$

where

d : denotes differential

t : time

x : n vector variable ("state of the system")

f : n vector function ("system equations")

ξ : n vector stochastic input the components of which are independent Wiener processes with zero mean and unit variance,

$$E[\{\xi(t) - \xi(\tau)\}\{\xi(t) - \xi(\tau)\}^T] = I \mid t - \tau \mid$$

$E[\cdot]$: expected value operator,

$\{\cdot\}$: column vector; superscript T denotes the transpose

$D(t)$: n by n matrix such that $D(t)\, D(t)^T = R(t)$, R being a positive semidefinite symmetric matrix (covariance).

Furthermore, let the observations "$y(t)$" on the process $x(t)$ be given by a set of stochastic differential equations interpreted in the sense of Ito:

$$dy = h(t, x)\, dt + N(t)\, d\eta, \tag{2}$$

where

y : m vector variable; $m \leqslant n$

h : m vector function

η : m vector stochastic input the components of which are independent Wiener processes with zero mean and unit variance

$$E[\{\eta(t) - \eta(\tau)\}\{\eta(t) - \eta(\tau)\}^T] = I \mid t - \tau \mid$$

and such that ξ and η are uncorrelated.

$N(t)$: m by m matrix such that

$$N(t)\, N(t)^T = Q(t),$$

Q being a positive definite symmetric matrix (covariance).

Let a continuous $z(t)$ process (called "filtering process") be also given by a set of stochastic differential equations (called "filter equations"):

$$dz = f(t, z)\, dt + G(t)\, [dy - h(t, z)\, dt], \tag{3}$$

where

z : n vector variable ("filtering estimate of x")

$G(t)$: n by m matrix ("filter gain").

The filter gain $G(t)$ in Eq. (3) is assumed to be given determined by some appropriate technique. Thus, two main assumptions are implied in Eq. (3): (a) the structure of the filter, (b) the character of the filter gain. These assumptions might be thought to be motivated by the form of some of the proposed suboptimal nonlinear filter algorithms [12–17] and by some experience [20–22]. In many cases, modern nonlinear filter algorithms are constructed and implemented according to Eq. (3) mainly because of constraints in mechanizing the filter algorithms.

Now, the question is to evaluate the performance characteristics of the nonlinear filter specified by Eq. (3). The common and only known method for that purpose is the application of Monte Carlo techniques. However, Monte Carlo techniques are usually very time consuming and expensive even for problems of moderate complexity and, even when the most advanced digital computers are utilized. Moreover, Monte Carlo techniques, being of numerical nature, can not provide concise and parametric answers regarding the filters' performance characteristics.

The aim of this article is to investigate the possibilities and ramifications of obtaining useful analytical methods for evaluating the performance of sub-optimal nonlinear filters specified by Eq. (3).

To pose the problem handy for analysis, Eq. (3) is now rewritten by combining it with Eq. (2):

$$dz = f(t, z)\, dt + G(t)\, [h(t, x) - h(t, z)]\, dt + G(t)\, N(t)\, d\eta. \tag{4}$$

The significance of Eq. (4) is the fact that it does not contain the observation variable "y" explicitly. Note that, in a Monte Carlo type performance evaluation procedure, the filter specified by Eq. (3) actually takes the form of Eq. (4) since the observation vector "y", based on the simulated solution of Eq. (1), is directly fed into Eq. (3).

Now, a joint $\{x(t), z(t)\}$ process can be defined by adjoining Eq. (4) to Eq. (1):

$$\begin{Bmatrix} dx \\ dz \end{Bmatrix} = \begin{Bmatrix} f(t, x) \\ f(t, z) + G(t)\, \{h(t, x) - h(t, z)\} \end{Bmatrix} dt + \begin{bmatrix} D(t) & 0 \\ 0 & G(t)\, N(t) \end{bmatrix} \begin{Bmatrix} d\xi \\ d\eta \end{Bmatrix}. \tag{5}$$

Obviously, the enlarged system of equations has $2n$ dimensions.

An error process $e(t)$ for the filtering process $z(t)$ is defined in a natural way by

$$e(t) = x(t) - z(t), \tag{6}$$

where $e(t)$ is clearly an n-vector. From Eqs. (5, 6), the time evolution of the filtering error process is governed by

$$\begin{aligned} de &= \{f(t, x) - f(t, z)\}\, dt - G(t)\, \{h(t, x) - h(t, z)\}\, dt \\ &\quad + D(t)\, d\xi - G(t)\, N(t)\, d\eta \\ &= k(t, x, z)\, dt + K(t)\, d\zeta, \end{aligned} \tag{7}$$

where

$$K(t)\, d\zeta \triangleq D(t)\, d\xi - G(t)\, N(t)\, d\eta$$

such that ζ is formally an equivalent n-vector stochastic input, the components of which are independent Wiener processes with zero mean and unit variance,

$$E[\{\zeta(t) - \zeta(\tau)\}\{\zeta(t) - \zeta(\tau)\}^T] = I \mid t - \tau \mid;$$

furthermore, $K(t)$ is formally an n by n matrix such that

$$K(t)\,K(t)^T = R(t) + G(t)\,Q(t)\,G(t)^T.$$

(Note that, by assumption, ξ and η are uncorrelated.)

The description of the stochastic characteristics of the filtering error process is in this investigation sought in terms of the mean $m(t)$ and covariance $B(t)$ of the $e(t)$ process defined by

$$m(t) = E[e(t)] \tag{8}$$

$$B(t) = E[\{e(t) - m(t)\}\{e(t) - m(t)\}^T]. \tag{9}$$

Now, we can also write

$$e(t) - m(t) = \{x(t) - \mu(t)\} - \{z(t) - \nu(t)\}, \tag{10}$$

where

$$\mu(t) = E[x(t)] \tag{10a}$$

$$\nu(t) = E[z(t)]. \tag{10b}$$

Thus, an alternative set of expressions for $m(t)$ and $B(t)$ can be written as

$$m(t) = \mu(t) - \nu(t) \tag{11}$$

$$\begin{aligned} B(t) &= E[\{\{x(t) - \mu(t)\} - \{z(t) - \nu(t)\}\}\{\{x(t) - \mu(t)\} - \{z(t) - \nu(t)\}\}^T] \\ &= C^{xx} + C^{zz} - C^{xz} - C^{zx}, \end{aligned} \tag{12}$$

where the superscripted matrices are defined as follows:

$$C^{xx} \triangleq E[\{x(t) - \mu(t)\}\{x(t) - \mu(t)\}^T] \tag{12a}$$

$$C^{zz} \triangleq E[\{z(t) - \nu(t)\}\{z(t) - \nu(t)\}^T] \tag{12b}$$

$$C^{xz} \triangleq E[\{x(t) - \mu(t)\}\{z(t) - \nu(t)\}^T] \tag{12c}$$

$$C^{zx} \triangleq E[\{z(t) - \nu(t)\}\{x(t) - \mu(t)\}^T]. \tag{12d}$$

The B, C^{xx} and C^{zz} matrices are symmetric.

However,

$$C^{xz} \neq C^{xz^T} \qquad C^{zx} \neq C^{zx^T} \tag{12e}$$

and, clearly,

$$C^{xz} \neq C^{zx}. \tag{12f}$$

But

$$C^{xz} = C^{zx^T} \quad \text{or} \quad C^{zx} = C^{xz^T}. \tag{12g}$$

Consequently, Eq. (12) can also be written as

$$B(t) = C^{xx} + C^{zz} - (C^{xz} + C^{xz^T}). \tag{12h}$$

Thus, there are two alternative routes for evaluating the performance of the filter specified by Eq. (3) in terms of the mean and covariance of the filtering error process: (i) starting from Eqs. (8, 9), (ii) starting from Eqs. (11, 12). It can be anticipated that the first approach is simpler (and possibly more natural) than the second one.

III. Markov Processes and the Kolmogorov Equations

The stochastic differential equation for the joint $\{x(t), z(t)\}$ process, Eq. (5), is in the form of the generalized Langevin equation. The stochastic differential equation for the $e(t)$ process, Eq. (7), will also be in the form of the generalized Langevin equation provided that the function $k(t, x, z)$ in Eq. (7) is expressed as $l(t, e)$. (This can be done approximately; cfr. next Section.) Thus, the solutions of Eq. (5) and Eq. (7) are random functions that are Markov processes [18, 19]. The Markov process hypothesis is a rather general and realistic model for a large class of processes of practical interest.

The conditional probability densities $p(\cdot \mid \cdot)$ and conditional probability distributions $P(\cdot \mid \cdot)$ for the $e(t)$ and $\{x(t), z(t)\}$ Markov processes, containing all relevant informations on these processes, are defined as

$$\begin{aligned}
p(t, e \mid t_0, e_0)\, de &\triangleq \Pr[e \leqslant e(t) < e + de \mid e(t_0) = e_0] \\
p(t, x, z \mid t_0, x_0, z_0)\, dx\, dz &\triangleq \Pr[x \leqslant x(t) < x + dx, z \leqslant z(t) \\
&\qquad < z + dz \mid x(t_0) = x_0, z(t) = z_0] \\
P(t, d\theta \mid t_0, e_0) &\triangleq p(t, \theta \mid t_0, e_0)\, d\theta \\
P(t, d\Omega \mid t_0, x_0, z_0) &\triangleq p(t, \Omega \mid t_0, x_0, z_0)\, d\Omega,
\end{aligned}$$

where, to simplify notations, the $e(t)$, $x(t)$, $z(t)$ processes are taken as scalar processes. (Note that the *joint* $\{x(t), z(t)\}$ process is a vector process even though $x(t)$ is a scalar process.) The time evolution of the conditional probability density and the conditional probability distribution of a Markov process satisfy the forward and backward Kolmogorov equations, respectively, under the usual analytic assumptions on the functions p and P[3, 6, 19]. The Kolmogorov equations in operator form are

$$\frac{\partial p}{\partial t} = L^+[p] \tag{13a}$$

$$\frac{\partial P}{\partial t_0} = L^-[P], \tag{13b}$$

where the forward L^+ and the backward L^- Kolmogorov operators (which are adjoint operators) are defined as follows:

$$L^+\odot = -\sum_i \frac{\partial}{\partial v_i}(\varphi_i(t, v)\odot) + \frac{1}{2}\sum_{i,j}\frac{\partial^2}{\partial v_i\,\partial v_j}(S_{ij}(t, v)\odot) \tag{14a}$$

$$L^-\odot = \sum_i \varphi_i(t, v)\frac{\partial}{\partial v_i}\odot + \frac{1}{2}\sum_{i,j} S_{ij}(t, v)\frac{\partial^2}{\partial v_i\,\partial v_j}\odot. \tag{14b}$$

The notations in Eqs. (14a–b) should be understood in terms of the generalized Langevin equation

$$dv = \varphi(t, v)\,dt + \omega(t, v)\,d\pi,$$

where φ is an n-vector function, π is an n-vector stochastic input, the components of which are independent Wiener processes with zero mean and unit variance and, $\omega(t, v)$ is an n by n matrix such that $\omega\omega^T = S$.

It can be shown (see Appendix A) that, in terms of the backward Kolmogorov operator L^-, the following equations can be written for the time derivative of the mean and covariance of the filtering error process:

Method I. Using Eqs. (8, 9) and the probability density of the $e(t)$ process,

$$\dot{m} = \int^n \cdots \int L^-\{e(t)\}\, p(t, e)\, de \tag{15}$$

$$\dot{B} = \int^n \cdots \int L^-[\{e(t) - m(t)\}\{e(t) - m(t)\}^T]\, p(t, e)\, de, \tag{16}$$

where the dot over a symbol denotes time derivative[1], L^- acts on the state variable "e" and, de is an infinitesimal element of the n-dimensional e-vector space.

[1] This notation for a time derivative is intended to signify in this paper that the respective time derivatives are to be taken in a *deterministic* sense.

Method II. Using Eqs. (11, 12) and the probability density of the joint $\{x(t), z(t)\}$ process,

$$\begin{Bmatrix} \dot{\mu} \\ \dot{\nu} \end{Bmatrix} = \int^{2n} \cdots \int L^{-} \begin{Bmatrix} x \\ z \end{Bmatrix} p(t, x, z)\, dx\, dz \tag{17}$$

$$\begin{aligned} \dot{B} &= \int^{2n} \cdots \int L^{-}[\{\{x(t) - \mu(t)\} - \{z(t) - \nu(t)\}\} \\ &\qquad \times \{\{x(t) - \mu(t)\} - \{z(t) - \nu(t)\}\}^T] \cdot p(t, x, z)\, dx\, dz \\ &= \dot{C}^{xx} + \dot{C}^{zz} - (\dot{C}^{xz} + \dot{C}^{xz^T}), \end{aligned} \tag{18}$$

where L^{-} acts on the enlarged state vector $\{x, z\}$ and, $dx\, dz$ is an infinitesimal element of the $2n$-dimensional $\{x, z\}$ vector space.

It is noted that, for an n-dimensional problem, Eqs. (15, 16) will involve $n(n + 3)/2$ ordinary differential equations, while Eqs. (17, 18) will involve $n(2n + 3)$ ordinary differential equations. Thus, for instance, for $n = 3$, Method I will require the solution of a system of 9 differential equations, while Method II will require the solution of a system of 27 differential equations. Clearly, the second task is much more complex than the first one.

To evaluate the integrals of Eqs. (15, 18) in analytical terms (that is, to obtain differential equations for the integro-differential equations given by Eqs. (15, 18)), some approximations are needed. The approximations will be introduced for Method I and Method II separately in order to provide deeper insight into the ramifications of the approximations.

IV. Approximations

Method I.

The operator L^{-} in Eqs. (15, 16) acts on the state variable "e". Thus, it is necessary to obtain a (stochastic) state differential equation for the $e(t)$ process in terms of "e" since Eq. (7), as it stays now, describes the $e(t)$ process in terms of the joint $\{x(t), z(t)\}$ process.

Performing a formal linear expansion of Eq. (7) about "e", cancelling and rearranging terms, one obtains the following stochastic dynamical system approximately describing the time evolution of the error process:

$$de = A(t, e)\, e\, dt + K(t)\, d\zeta, \tag{19}$$

where

$$A(t, e) = f'(t, e) - G(t)\, h'(t, e) \tag{19a}$$

and the prime (′) denotes the Jacobian of the respective vectors taken at "e". For instance, $f'(t, e) = (\partial f_i/\partial x_j)_e$.

To illuminate the steps involved in deriving the differential equations for the mean $m(t)$ and covariance $B(t)$ of the filtering error process, the scalar case is elaborated first.

Scalar case.

Since now every term in Eq. (19) is a scalar quantity, performing the L^- operation in Eq. (15) yields

$$\dot{m} = \int a(t, e)\, ep(t, e)\, de, \tag{20}$$

where

$$a(t, e) = f'(t, e) - g(t)\, h'(t, e) \tag{21}$$

with $g(t)$: scalar filter gain, $f'(t, e) = (\partial f/\partial x)_e$, etc.

Expanding $a(t, e)$ about "m" up to linear term in $(e - m)$,

$$\dot{m} \simeq \int e[a(t, m) + a'(t, m)\,(e - m) + \cdots +]\, p(t, e)\, de$$

and performing the expectation integrals, one obtains

$$\dot{m} = a(t, m)\, m + a'(t, m)\, b(t), \tag{22}$$

where $b(t)$ denotes the variance of the (scalar) filtering error and the prime (′) means derivative with respect to "e" taken at "m", i.e.,

$$a'(t, m) = f''(t, m) - g(t)\, h''(t, m). \tag{23}$$

The initial condition for Eq. (22) is $m(0) = 0$. This is so since $z(0) = E[x(0)]$ is used in filtering theory such that $E[x(0)]$ belongs to the given part of the filtering problem. Hence, $E[z(0)] = E[x(0)]$ yielding $m(0) = 0$.

Eq. (22) with $m(0) = 0$ approximately describes the time evolution of the mean of the filtering error process for a filter specified by Eq. (3). It is noted that, for linear problems, $a'(t, m) \equiv 0$ which, according to Eq. (22) and $m(0) = 0$, results in $m(t) \equiv 0$. Furthermore, in such nonlinear cases if

$$a'(t, m) = \alpha(t, m)\, m, \tag{24}$$

then Eq. (22) becomes

$$\dot{m} = [a(t, m) + \alpha(t, m)\, b(t)]\, m \tag{25}$$

yielding again $m(t) \equiv 0$ since $m(0) = 0$. In the most general case, however, the filtering error variance $b(t)$ remains *effectively* coupled to the differential equation for the mean of the filtering error as it is given by Eq. (22).

Now, performing the L^- operation in Eq. (16) and considering Eq. (19), one finds:

$$\dot{b} = \int [2a(t, e)\, e(e - m) + g^2(t)\, q^2(t) + r^2(t)]\, p(t, e)\, de \tag{26}$$

where q^2 and r^2 are the variances of the "Gaussian white" measurement and dynamical noise, respectively.

Again, taking a linear expansion of $a(t, e)$ about "m", one finds for Eq. (26)

$$\begin{aligned} \dot{b} \simeq \int \{&2e(e - m)\,[a(t, m) + a'(t, m)\,(e - m) + \cdots +] \\ &+ g^2(t)\, q^2(t) + r^2(t)\}\, p(t, e)\, de. \end{aligned} \tag{27}$$

Performing the expectation integral in Eq. (27), it is observed that the term belonging to $a'(t, m)$ would yield moments higher than second order. Because of our aim to find analytical means for evaluating the filter's performance up to second-order statistics, the $a'(t, m)$ term will be neglected in Eq. (27). We then obtain the following differential equation approximately describing the time evolution of the variance of the filtering error process

$$\dot{b} = 2a(t, m)\, b + g^2(t)\, q^2(t) + r^2(t), \tag{28}$$

with initial condition $b(0) = b_0$ given such that

$$b_0 = E[\{z(0) - \mu(0)\}^2] = E[\{x(0) - \nu(0)\}^2],$$

where the last quantity belongs to the given part of the filtering problem.

In general, Eqs. (22) and (28) which are the desired results in this case, form a *coupled* system of (deterministic) differential equations approximately describing the performance of the filter specified by Eq. (3) in terms of the mean and variance of the filtering error process and in terms of known system and noise parameters and filter gain. However, when Eq. (24) holds, then $m(t) \equiv 0$ which in turn yields that Eq. (28) becomes decoupled from Eq. (22):

$$\dot{b} = 2[f'(t, 0) - g(t)\, h'(t, 0)]\, b + g^2(t)\, q^2(t) + r^2(t). \tag{29}$$

This equation can be integrated by itself for the given initial condition b_0. It is noted that Eq. (29) is *exact* for linear problems. Note also that Eq. (29) will hold for nonlinear problems not only when Eq. (24) holds but also when the $a'(t, m)$ term containing the second derivatives of f and h is deliberately neglected in Eq. (22). Neglecting the $a'(t, m)$ term in Eq. (22) can be justifiable in a number of cases.

Vector case

Deriving differential equations for the mean and covariance of the filtering error process in the vector case essentially involves the same steps and the same type of approximations as those outlined above in the scalar case. Taking proper care of the vector and matrix algebra involved in the L^- operator and in the multidimensional Taylor expansions, one obtains

$$\dot{m} = A(t, m)\, m + \{f''_m : B\} - G\{h''_m : B\} \tag{30}$$

$$\dot{B} = A(t, m)\, B + BA^T(t, m) + G(t)\, Q(t)\, G^T(t) + R(t) \tag{31}$$

with

$$A(t, m) = f_m' - Gh_m',$$

where f_m' and h_m' denote the Jacobians of the vector functions f and h taken at "m." Furthermore, $\{f''_m : B\}$ and $\{h''_m : B\}$ denote column vectors such that, e.g.,

$$\{f''_m : B\} = \begin{Bmatrix} \mathrm{Tr}[f''_{1_m} B] \\ \vdots \quad \vdots \\ \mathrm{Tr}[f''_{n_m} B] \end{Bmatrix},$$

where the i-th component is given by

$$\mathrm{Tr}[f''_{i_m} B] = \frac{1}{2} \sum_{j,k}^{n} \left(\frac{\partial^2 f_i}{\partial x_j\, \partial x_k} \right)_m B_{jk}\,.$$

Thus, the notation f''_{i_m} means the Hessian matrix of the multivariate scalar function f_i taken at "m."

The initial conditions for Eqs. (30, 31) are $m(0) = 0$, $B(0) = B_0$ given, using the same explanation as in the scalar case.

In general, Eqs. (30, 31) which are the desired results in this case, form a coupled system of $n(n + 3)/2$ ordinary nonlinear differential equations approximately describing the mean error and covariance of the filter specified by Eq. (3) for an n-dimensional process. Eqs. (30, 31) are in terms of known system and noise parameters and filter gain and, can be integrated simultaneously for given initial conditions. However, when the last two terms in Eq. (30) can be expressed as

$$\{f''_m : B\} - G\{h''_m : B\} = \Psi(B, m)\, m, \tag{32}$$

then Eq. (30) yields $m(t) \equiv 0$ since $m(0) = 0$. Consequently, Eq. (31) can be integrated by itself since then

$$\dot{B} = A(t, 0)\, B + BA^T(t, 0) + G(t)\, Q(t)\, G^T(t) + R(t) \tag{33}$$

describes B independently of m. Clearly, Eq. (33) is exact for linear problems since Eq. (32) is identically zero. Note that Eq. (33) also holds when terms containing second derivatives of f and h are deliberately neglected in Eq. (30). This can be justifiable in several cases.

Method II

In this approach, the aim is to evaluate the integrals of Eqs. (17, 18). Since the joint $\{x(t), z(t)\}$ process is explicitly described by Eq. (5) in terms of $\{x, z\}$, the integrals in Eqs. (17, 18) can be evaluated immediately without making any approximation on the state description itself. (This is the main difference between the approach of Method I and Method II. In Method I, an approximate state description had first to be developed for the $e(t)$ process in order to evaluate the integrals of Eqs. (15, 16).)

Again, to illuminate the steps to derive approximate differential equations for the mean and covariance of the filtering error process, the scalar case is treated first.

Scalar case

Though the $x(t)$ and $z(t)$ processes are scalar processes, the joint $\{x(t), z(t)\}$ process is a 2-dimensional vector process. Thus, the L^- operator in Eqs. (17, 18) has to be interpreted as a vector differential operator even for a scalar $z(t)$ process.

Performing the L^- operation in Eq. (17) yields

$$\begin{Bmatrix} \dot{\mu} \\ \dot{\nu} \end{Bmatrix} = \iint \begin{Bmatrix} f(t, x) \\ f(t, z) + g(t)\,[h(t, x) - h(t, z)] \end{Bmatrix} p(t, x, z)\, dx\, dz, \tag{34}$$

where $g(t)$ is the scalar filter gain, and $p(t, x, z)$ is the joint probability density of the $\{x(t), z(t)\}$ process. Now, expanding $f(t, x)$, $h(t, x)$ about μ and $f(t, z)$, $h(t, z)$ about ν up to *second*-order terms and performing the expectation integrals, the following differential equations are obtained from Eq. (34) for the components of the mean filtering error:

$$\dot{\mu} = f(t, \mu) + \tfrac{1}{2} f''(t, \mu)\, c_{11}(t) \tag{35}$$

$$\begin{aligned} \dot{\nu} = {} & f(t, \nu) + \tfrac{1}{2} f''(t, \nu)\, c_{22}(t) + g(t)\,[h(t, \mu) - h(t, \nu)] \\ & + \tfrac{1}{2} g(t)\,[h''(t, \mu)\, c_{11}(t) - h''(t, \nu)\, c_{22}(t)], \end{aligned} \tag{36}$$

where double prime (″) denotes second derivative with respect to the state, c_{11}: variance of the $x(t)$ process by itself, and c_{22}: variance of the $z(t)$ process by itself. (Cf. Eqs. (12a–b) for the vector case.)

Performing the L^- operation in Eq. (18) yields the following component equations for the time evolution of the variance of the scalar filtering error process:

$$\dot{c}_{11} = \iint [2(x-\mu)f(t,x) + r^2(t)]\, p(t,x,z)\, dx\, dz \tag{37a}$$

$$\dot{c}_{12} = \dot{c}_{21} = \iint \{(z-\nu)f(t,x) + (x-\mu)\,[f(t,z) + g(t)\,[h(t,x) - h(t,z)]]\} \cdot p(t,x,z)\, dx\, dz \tag{37b}$$

$$\dot{c}_{22} = \iint \{2(z-\nu)\,[f(t,z) + g(t)\,[h(t,x) - h(t,z)] + g^2(t)\, q^2(t)\}] \times p(t,x,z)\, dx\, dz, \tag{37c}$$

where r^2 and q^2 are the variance, respectively, of the "Gaussian white" system and measurement noise, and $c_{12} = c_{21}$ is the covariance of the joint $\{x(t), z(t)\}$ process. (Cf. Eqs. (12c, d) for the vector case.) Note, there is no additional "forcing term" in Eq. (37b) since, by assumption, the system and measurement noise (ξ, η) are uncorrelated.

Now, expanding $f(t,x)$, $h(t,x)$ about μ and $f(t,z)$, $h(t,z)$ about ν up to *first*-order terms and so performing the expectation integrals, the following differential equations are obtained from Eqs. (37a–c):

$$\dot{c}_{11} = 2f'(t,\mu)\, c_{11} + r^2(t) \tag{38a}$$

$$\dot{c}_{12} = \dot{c}_{21} = [f'(t,\mu)\, c_{11} + f'(t,\nu)]\, c_{12} + g(t)\,[h'(t,\mu)\, c_{11} - h'(t,\nu)\, c_{12}] \tag{38b}$$

$$\dot{c}_{22} = 2f'(t,\nu)\, c_{22} + 2g(t)\,[h'(t,\mu)\, c_{12} - h'(t,\nu)\, c_{22}] + g^2(t)\, q^2(t). \tag{38c}$$

Note, that an expansion in Eqs. (37a–c) higher than the first order would result moments higher than the second order when the expectation integrals are performed. This, in turn, would violate our basic aim to evaluate the filters' performance in terms of second-order statistics only.

Eqs. (35, 36) together with Eqs. (38a–c) are the desired results in this case. They form a *coupled* system of 5 ordinary nonlinear differential equations approximately describing the time evolution of the mean,

$$\dot{m} = \dot{\mu} - \dot{\nu} \tag{39}$$

and the time evolution of the variance,

$$\dot{b} = \dot{c}_{11} + \dot{c}_{22} - 2\dot{c}_{12} \tag{40}$$

of the filtering error process for the filter specified by Eq. (3). Eqs. (35–36) and Eqs. (38a–c) are in terms of known system and noise parameters and filter gain and, can be integrated simultaneously in a deterministic sense once their initial conditions are specified.

Following common practice in modern filtering theory, it seems reasonable to assume that the apriori distributions for the $x(t)$ and $z(t)$ processes are identical. That is,

$$E[z(0)] = E[x(0)] = \mu_0 \tag{41a}$$

$$E[\{z(0) - \mu(0)\}^2] = E[\{x(0) - \nu(0)\}^2] = c_{11_0}, \tag{41b}$$

where the right sides in Eqs. (41a, b) are assumed to be given as part of the filtering problem. Thus, the initial conditions for Eqs. (35, 36) and Eqs. (38a–c) are specified as follows:

$$\nu_0 = \mu_0 \tag{42a}$$

$$c_{22_0} = c_{11_0} \tag{42b}$$

$$c_{12_0} = \tfrac{1}{2} c_{11_0}, \tag{42c}$$

where the last condition follows from the fact that we also have

$$b_0 = c_{11_0} \qquad \text{and} \qquad b_0 = c_{11_0} + c_{22_0} - 2c_{12_0}.$$

When Eqs. (39, 40) are written out fully, we have

$$\begin{aligned} \dot{m} &= [f(t, \mu) - g(t)\, h(t, \mu)] - [f(t, \nu) - g(t)\, h(t, \nu)] \\ &\quad + \tfrac{1}{2}[f''(t, \mu) - g(t)\, h''(t, \mu)]\, c_{11} - \tfrac{1}{2}[f''(t, \nu) - g(t)\, h''(t, \nu)]\, c_{22} \end{aligned} \tag{43}$$

$$\begin{aligned} \dot{b} &= 2[f'(t, \nu) - g(t)\, h'(t, \mu)]\,(c_{11} - c_{12}) \\ &\quad + 2[f'(t, \nu) - g(t)\, h'(t, \nu)]\,(c_{22} - c_{12}) + g^2(t)\, q^2(t) + r^2(t). \end{aligned} \tag{44}$$

In these equations, $\dot{m} = \varphi(\mu, \nu, c_{11}, c_{22})$, $\dot{b} = \Psi(\mu, \nu, c_{11}, c_{12}, c_{22})$. It is interesting to note that Eqs. (43, 44) can *not* be reduced to the simultaneous form $\dot{m} = \varphi(m, b)$ and $\dot{b} = \Psi(m, b)$ like the form of Eqs. (22) and (28) of Method I. This is significant since Eqs. (43, 44) of Method II and Eqs. (22) and (28) of Method I are on the same level of approximations as far as the expansions for the f and h functions are concerned. This fact clearly illustrates that the two methods have different avenues.

However, when the functions f and h in Eqs. (43, 44) are formally expanded about m such that second derivatives of f and h are neglected (which, in general, might be regarded as a very bold approximation), one finds:

$$\dot{m} = [f'(t, m) - gh'(t, m)]\, m \tag{45}$$

$$\dot{b} = 2[f'(t, m) - g(t)\, h'(t, m)]\, b + g^2(t)\, q^2(t) + r^2(t). \tag{46}$$

Note, that the approximation leading to Eqs. (45, 46) means that μ and ν are simply replaced by "m" in Eq. (44) and, the c_{11} and c_{22} terms are entirely neglected in Eq. (43). (Note also that Eqs. (45, 46) are not the same as Eq. (22) and Eq. (28) of Method I since now Eq. (45) is decoupled from Eq. (46).) The approximations involved in arriving to Eqs. (45, 46) are fully justified, however, when the second derivatives of f and h can be expressed in the form of Eq. (24). In that case $m(t) \equiv 0$ since $m(0) = 0$, even if second derivatives of f and h are retained in the expansion of Eqs. (43, 44). When $m(t) \equiv 0$, then we also have

$$\dot{b} = 2[f'(t, 0) - g(t)\, h'(t, 0)]\, b + g^2(t)\, q^2(t) + r^2(t), \tag{47}$$

which is identical with Eq. (29) of Method I and can be integrated by itself for given $b_0 = c_{11_0}$. It is also noted that Eqs. (45–47) are exact equations for linear problems.

For the general case, however, the features of Method I and Method II should be compared by comparing Eqs. (22) and (28) of Method I to Eqs. (35, 36) and Eqs. (38a–c) of Method II. Determining $m(t)$ and $b(t)$ from Eqs. (35, 36) and (38a–c) is computationally a much more complex task than the simultaneous integration of Eq. (22) and Eq. (28).

Vector case

The derivation of (deterministic) differential equations governing the time evolution of the component vectors and component matrices of the mean and covariance of the filtering error process in the case of an n-dimensional $x(t)$ process essentially involves the same procedures and approximations as those applied above in the scalar case. Taking proper care of the vector and matrix operations involved in the L^- operator and in the expansions (which requires rather extensive algebraic manipulations), one obtains

$$\dot{\mu} = f(t, \mu) + \{f''_\mu : C^{xx}\} \tag{48}$$

$$\begin{aligned} \dot{\nu} &= f(t, \nu) + G\{h(t, \mu) - h(t, \nu)\} \\ &\quad + \{f''_\nu : C^{zz}\} + G\{h''_\mu : C^{xx}\} - G\{h''_\nu : C^{zz}\} \end{aligned} \tag{49}$$

$$\dot{C}^{xx} = f_u' C^{xx} + C^{xx}[f_u']^T + R \tag{50a}$$

$$\begin{aligned} \dot{C}^{zz} = {} & [f_\nu' - Gh_\nu'] \, C^{zz} + C^{zz}[f_\nu' - Gh_\nu']^T \\ & + Gh_\mu'[C^{zx}]^T + C^{zx}[Gh_\mu']^T + GQG^T \end{aligned} \tag{50b}$$

$$\dot{C}^{zx} = f_\nu' C^{zx} + C^{zx}[f_\mu]^T + G[h_\mu' C^{xx} - h_\nu' C^{zx}] \tag{50c}$$

$$\dot{C}^{xz} = [\dot{C}^{zx}]^T, \tag{50d}$$

where the notations have the same meaning as described previously (cf. the text following Eq. (31), but now, for easier reading, the notation of time-dependency is omitted in most of the terms of Eqs. (48–50d)).

In general, Eqs. (48–50c) which are the desired results in this case, form a coupled system of $n(2n + 3)$ ordinary nonlinear differential equations approximately describing the *components* of the mean error and covariance of the filter specified by Eq. (3) for an n-dimensional process in terms of known system and noise parameters and filter gain. Reasoning similarly as in the scalar case above, the initial conditions for Eqs. (48–50c) are specified as follows:

$$\nu_0 = \mu_0 \tag{51a}$$

$$C_0^{zz} = C_0^{xx} \tag{51b}$$

$$C_0^{zx} = \tfrac{1}{2} \, C_0^{xx}, \tag{51c}$$

where μ_0 and C_0^{xx} belong to the given part of the filtering problem. (Note, that Eq. (51c) results a symmetric apriori distribution for the nonsymmetric matrix C^{zx}.)

The equations for $\dot{m}$ and $\dot{B}$ resulting from Eqs. (48–50c) are the following:

$$\begin{aligned} \dot{m} &= \dot{\mu} - \dot{\nu} \\ &= [f(t, \mu) - G(t) \, h(t, \mu)] - [f(t, \nu) - G(t) \, h(t, \nu)] \\ &\quad + \{f_\mu'' : C^{xx}\} - \{f_\nu'' : C^{zz}\} - G\{h_\mu'' : C^{xx}\} + G\{h_\nu'' : C^{zz}\} \end{aligned} \tag{52}$$

$$\begin{aligned} \dot{B} &= \dot{C}^{xx} + \dot{C}^{zz} - (\dot{C}^{zx} + [\dot{C}^{zx}]^T) \\ &= [f_\mu' - Gh_\mu'] \, [C^{xx} - [C^{zx}]^T] + [C^{xx} - C^{zx}] \, [f_\mu' - Gh_\mu']^T \\ &\quad + [f_\nu' - Gh_\nu'] \, [C^{zz} - C^{zx}] + [C^{zz} - [C^{zx}]^T] \, [f_\nu' - Gh_\nu']^T \\ &\quad + GQG^T + R. \end{aligned} \tag{53}$$

To determine $m(t)$ and $B(t)$, Eqs. (52, 53) are useless unless the right side of these equations is expressed in terms of m and B. This can be achieved (like in the scalar case) only by rejecting second derivatives of f and h when Eqs. (52, 53) are formally expanded about "m." (Note again that, in doing so, the f'' and h'' terms in Eq. (52) are automatically rejected making Eq. (52) decoupled from Eq. (53).) This procedure yields the exact equations for linear problems

$$\dot{m} = [f_m' - Gh_m']\, m \tag{54}$$

$$\dot{B} = [f_m' - Gh_m']\, B + B[f_m' - Gh_m']^T + GQG^T + R. \tag{55}$$

However, for nonlinear problems Eqs. (54, 55) are also valid approximations provided that the second derivative terms of f and h in Eqs. (52, 53) can be expressed in a form similar to Eq. (32) since then $m(t) \equiv 0$. If so, then Eq. (33) is also a valid approximation. (Note, that Eq. (33) is exact for linear problems.)

In general, however, the features of the results of Method I and Method II should be compared by comparing Eqs. (30, 31) to Eqs. (48–50c). Again, it is noted that, from a computational point of view, it is much more attractive to determine $m(t)$ and $B(t)$ from Eqs. (30, 31) than by integrating Eqs. (48–50c) and so computing $m(t)$ and $B(t)$ from Eq. (11) and Eq. (12h).

V. Remarks

1. *Linear problems*

It was pointed out in each approximation procedure that, for linear problems, (a) both Methods yield the same differential equation (Eq. (29) or (33) for scalar or vector problems, respectively) describing the time evolution of the filtering error covariance, and (b) this differential equation is exact and valid for any filter gain of the form $g(t)$ or $G(t)$.

Now, Eqs. (29) and (33) establish a state differential equation relationship for the filtering error covariance in terms of known system and noise parameters and filter gain. Considering the gain function as a "control input" in Eq. (29) or (33), a well-defined deterministic optimization problem can be posed: Given Eq. (29) or (33) with known initial conditions, find the filter gain that minimizes the "performance index" J expressed as an appropriate scalar measure on the filtering error covariance.

For scalar problems, the optimization requires simple calculations. Introducing

$$J = \underset{g(t)}{\mathrm{Min}}[b(\tau)], \tag{56}$$

where τ is an unspecified terminal time and, for instance, applying the Pontryagin Maximum Principle, one finds that the Hamiltonian

$$H \triangleq b(\tau) + \lambda\{2[f(t) - g(t)\, h(t)]\, b + g^2(t)\, q^2(t) + r^2(t)\} \tag{57}$$

is minimized by

$$\lambda\{-2h(t)\,b(t) + 2q^2(t)\,g(t)\} = 0 \tag{58}$$

yielding

$$g^*(t) = \frac{1}{q^2(t)}\,h(t)\,b^*(t) \tag{59}$$

as the optimal (linear) filter gain that minimizes the filtering error variance for any time. Not unexpectedly, this optimal filter gain is identical to the Kalman–Bucy minimum variance linear filter gain [1]. Substituting Eq. (59) into Eq. (29) results in the Riccati differential equation governing the time history of the variance of the optimal linear filter. (This filter is optimal for a large class of performance indices, as it is easily seen from the Hamiltonian.)

For vector problems, a similar optimization procedure (of course, requiring more extensive algebra) yields the optimal matrix gain

$$G^*(t) = B^*(t)\,H^T(t)\,Q^{-1}(t) \tag{60}$$

which, substituted into Eq. (33), results in the matrix Riccati differential equation for the time evolution of the filtering error covariance of the optimum linear filter. This is, of course, again identical to the results of the Kalman–Bucy optimal linear filter theory [1].

2. *Deterministic filter gains for nonlinear problems*

In the performance analysis of nonlinear filters specified by Eq. (3) it was assumed that the filter gain, $g(t)$ or $G(t)$, is given as part of the problem. The derived (deterministic) differential equations for the mean and covariance of the filtering error are in terms of known system and noise parameters and filter gain. Thus, considering the filter gain as a "control input" in the derived differential equations (in a similar manner as in the linear problems above), and imposing suitable requirements on the filters' performance, appropriate deterministic filter gains can be determined for nonlinear filters specified by Eq. (3).

The simplest and most straightforward way of determining appropriate deterministic filter gains for nonlinear problems arises when the differential equation for the filtering error covariance $B(t)$ is decoupled from the differential equation for the mean filtering error $m(t)$ such that $m(t) \equiv 0$. (This can be considered as an approximation by itself or as a property of a class of nonlinear problems when $\dot{m} = \Psi(m, B)\,m$.) If so, then Eq. (29) or (33) can be treated exactly in the same way as in the linear problems above. That is, a Kalman–Bucy type filter gain can be easily predetermined by solving a Riccati differential equation. Clearly, the same Riccati differential equation will also provide an approximate description for the time evolution of the filtering error covariance.

Note that the above described way of applying the Riccati differential equation to nonlinear filtering problems when the filters' structure is specified by Eq. (3) is essentially different from the method of linearizing the nonlinear problem and so applying the Kalman–Bucy linear filter algorithm by using the estimated values of the state vector to compute the linear perturbation coefficients in the filter. In the latter case, the filter gain is a *stochastic* quantity and the Riccati equation provides a *stochastic* description of the filters' performance. While the methods of this paper imply that the filter gain and the Riccati equation are of *deterministic* nature.

These and other features of the analytical results of this paper are illustrated by two examples in the subsequent Section. (Several comparative case studies will be published elsewhere [23].)

VI. Examples

1. *A scalar problem*

Let the system and observations be given by

$$dx = -\frac{x}{1+x^2}\,dt + d\xi \tag{61}$$

$$dy = \begin{cases} x\,dt + d\eta & \text{(62a)} \\ \arctan(x)\,dt + d\eta & \text{(62b)} \end{cases}$$

where ξ and η are independent Wiener processes characterized by zero mean and standard deviation σ_s and σ_m, respectively. The initial condition on Eq. (61) is also specified in terms of a normal distribution $N(\alpha, \sigma_0)$ with given mean α and standard deviation σ_0. Hence, $x(0) = \alpha$.

For convenience, let a constant filter gain be postulated and denoted by γ. Thus, according to the structure of Eq. (3), the following nonlinear filters are constructed:

$$dz = -\frac{z}{1+z^2}\,dt + \gamma[dy - z\,dt] \tag{63a}$$

and

$$dz = -\frac{z}{1+z^2}\,dt + \gamma[dy - \arctan(z)\,dt] \tag{63b}$$

for linear and nonlinear measurements, respectively (z denotes the filtering estimate of x). The initial condition for Eqs. (63a, b) is $z(0) = x(0) = \alpha$.

In this case, the differential equation for the mean filtering error is in the form $\dot{m} = \varphi(m, b)\,m$ for the linear and the nonlinear measurements as well, yielding $m(t) \equiv 0$ since $m(0) = 0$. Thus, the variance equation is in the form of Eq. (29), i.e.,

$$\dot{b} = -2(1+\gamma)\, b + \gamma^2\sigma_m^2 + \sigma_s^2 \tag{64}$$

with $b_0 = \sigma_0^2$. Eq. (64) has a solution in closed form

$$b = \bar{b} + (\sigma_0^2 - \bar{b}) \exp[-2(1+\gamma)\, t], \tag{65}$$

where

$$\bar{b} = \frac{\sigma_s^2 + \gamma^2\sigma_m^2}{2(1+\gamma)} \tag{66}$$

is the steady state ($t \to \infty$) variance. The value of the constant filter gain γ^* that minimizes the steady state variance can easily be found from $d\bar{b}/d\gamma = 0$ resulting in

$$\gamma^* = \left[1 + \left(\frac{\sigma_s}{\sigma_m}\right)^2\right]^{\frac{1}{2}} - 1. \tag{67}$$

This in turn yields for the minimum steady state variance

$$\bar{b}^* = \sigma_m^2\gamma^*. \tag{68}$$

Figures 1 and 2 display some computed cases comparing the analytically predicted values to the "experimentally" determined values of the filtering error variance. "Experimental determination" means Monte Carlo simulation of the filter on the digital computer.[2]

Fig. 1 depicts the results for two sets of noise parameters with constant filter gains determined from Eq. (67). ("Minimum Steady State Variance Constant Gains".) Fig. 2 depicts the results for one fixed set of noise parameters but with two different constant gains: one determined from Eq. (67) and one picked up arbitrarily. The "theoretical" curves on Figs. 1 and 2 are the solutions of Eq. (64) with the relevant constant gains and noise parameters. As seen in Figs. 1 and 2, the "experimental" values do indeed converge to the analytically predicted values of the filtering error variance, and the steady state "experimental" and analytical values agree completely. Minor

[2] In the Monte Carlo simulations, the sampling time was made equal to the integration step size Δt which was 0.1 sec. In the numerical integration scheme (third order Runge–Kutta–Gill algorithm) the digitally generated Gaussian random numbers were utilized in the sense of a Markov *sequence* such that the specified random *process* standard deviation σ was translated into a random *sequence* standard deviation σ' through the equivalence claim $\sigma' = \sigma/\sqrt{\Delta t}$. The sequential random numbers were held constant during Δt. The sample space for determining the "experimental" values of the filtering error mean and variance contained 360 integration runs for each Monte Carlo simulation study. In the figures the simulation results are depicted in 0.5 sec intervals.

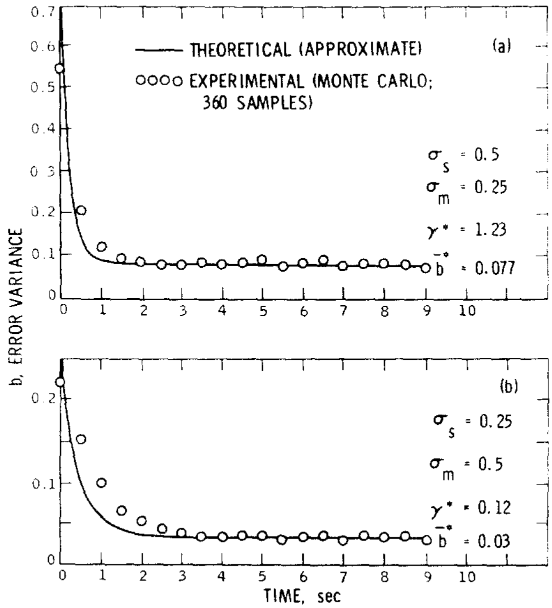

FIG. 1. Method I (scalar problem).

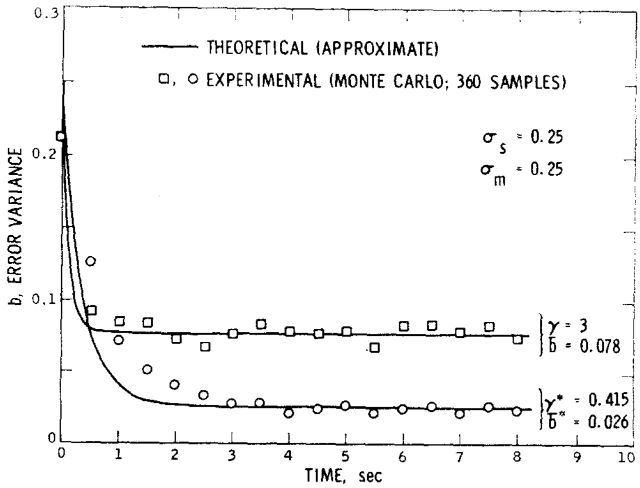

FIG. 2. Method I (scalar problem).

differences between the "experimental" and analytical values of the *transient* variance—as Figs. 1 and 2 display it—are expected, however, because of the simplifying assumptions involved in deriving the approximate analytical equations for the filtering error variance.

To illustrate the technical features of Method II, Fig. 3 depicts the results of the same scalar problem computed by Method I and displayed in Figure 1a, but now computed by Method II. The relevant differential equations derived by Method II for this particular problem are compiled in Appendix B. Comparing the solid curve of Fig. 3b to the solid curve of Fig. 1a, it is clear that the two Methods yield identical results. The avenues of the two Methods, however, are different. Clearly, Method II requires more labor (but, eventually, it also provides more detailed information) than Method I.

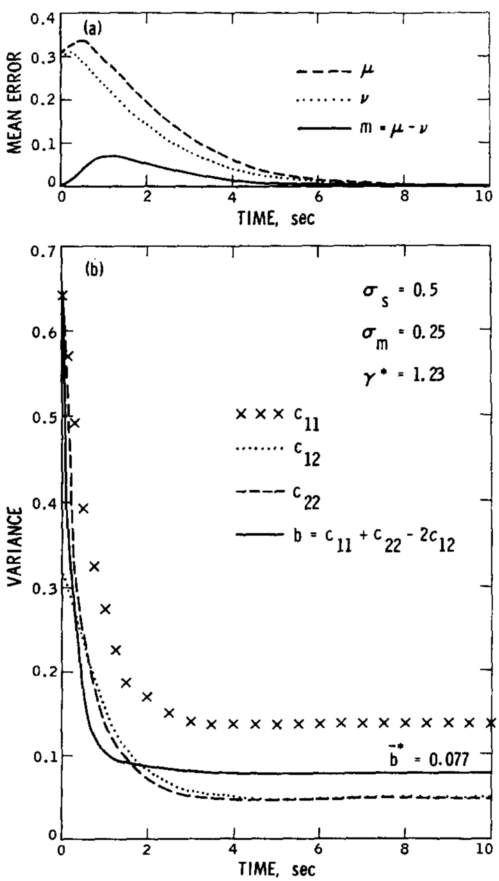

FIG. 3. Method II (scalar problem).

2. *A vector problem*

Let the system (a nonlinear forced spring) and observations be given by

$$\begin{Bmatrix} dx_1 \\ dx_2 \end{Bmatrix} = \begin{Bmatrix} x_2 \\ -2x_1 - ax_1^3 - 3x_2 \end{Bmatrix} dt + \begin{Bmatrix} d\xi_1 \\ d\xi_2 \end{Bmatrix} \tag{69}$$

$$\begin{Bmatrix} dy_1 \\ dy_2 \end{Bmatrix} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} dt + \begin{Bmatrix} d\eta_1 \\ d\eta_2 \end{Bmatrix}, \tag{70}$$

where $a = 0.5$ and, ξ_i and η_i are mutually independent Wiener process characterized by zero mean and covariance

$$R = \begin{bmatrix} R_{11} & 0 \\ 0 & R_{22} \end{bmatrix} \qquad Q = \begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{bmatrix}$$

for the system and measurement noise, respectively. The initial conditions for Eq. (69) are also specified in terms of normal distributions, $x_1(0) : N(\alpha_1, \sigma_{1_0})$, $x_2(0) : N(\alpha_2, \sigma_{2_0})$, with given values for the mean α_i and standard deviation σ_{i_0}. Hence, $x_1(0) = \alpha_1$, $x_2(0) = \alpha_2$.

Again, a constant filter gain is postulated and denoted by Γ. Thus, according to the structure of Eq. (3), the following nonlinear filter is constructed:

$$\begin{Bmatrix} dz_1 \\ dz_2 \end{Bmatrix} = \begin{Bmatrix} z_2 \\ -2z_1 - az_1^3 - 3z_2 \end{Bmatrix} dt + \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix} \begin{Bmatrix} dy_1 - z_1\,dt \\ dy_2 - z_2\,dt \end{Bmatrix}, \tag{71}$$

with initial conditions $z_1(0) = \alpha_1$, $z_2(0) = \alpha_2$. (Again, z_i denotes the estimated value of x_i.)

For this problem, the differential equation for the mean filtering error is in the form $\dot{m} = \Psi(m, B)\, m$. Hence, $m(t) \equiv 0$ since $m(0) = 0$. Thus, the variance equation is in the form of Eq. (33). That is,

$$\dot{B}_{11} = -2\Gamma_{11}B_{11} + 2(1 - \Gamma_{12})\,B_{12} + Q_{11}\Gamma_{11}^2 + Q_{22}\Gamma_{12}^2 + R_{11} \tag{72a}$$

$$\begin{aligned} \dot{B}_{12} = &-(2 + \Gamma_{12})\,B_{11} - (3 + \Gamma_{11} + \Gamma_{22})\,B_{12} \\ &+ (1 - \Gamma_{12})\,B_{22} + Q_{11}\Gamma_{11}\Gamma_{21} + Q_{22}\Gamma_{12}\Gamma_{22} \end{aligned} \tag{72b}$$

$$\dot{B}_{22} = -2(2 + \Gamma_{21})\,B_{12} - 2(3 + \Gamma_{22})\,B_{22} + Q_{11}\Gamma_{21}^2 + Q_{22}\Gamma_{22}^2 + R_{22} \tag{72c}$$

with given initial conditions

$$B_{11}(0) = \sigma_{1_0}^2, \qquad B_{12}(0) = 0, \qquad B_{22}(0) = \sigma_{2_0}^2.$$

In the computations, the constant gain matrix Γ^* will be applied that minimizes the trace of the steady state value $\bar{B}^*$ of the covariance. This requirement yields the same construction for the gain matrix as it is given by Eq. (60) for the optimal linear Kalman–Bucy filter except that now the value of $\bar{B}^*$ is applied in Eq. (60). The value of $\bar{B}^*$ can be determined by substituting

$$\begin{bmatrix} \Gamma_{11}^* & \Gamma_{12}^* \\ \Gamma_{21}^* & \Gamma_{22}^* \end{bmatrix} = \begin{bmatrix} \bar{B}_{11}^* Q_{11}^{-1} & \bar{B}_{12}^* Q_{22}^{-1} \\ \bar{B}_{12}^* Q_{11}^{-1} & \bar{B}_{22}^* Q_{22}^{-1} \end{bmatrix} \tag{73}$$

into the steady state version of Eqs. (72a–c) and solving the resulting *algebraic* (Riccati) system of equations for $\bar{B}_{ij}^*$ (To find $\bar{B}_{ij}^*$, however, it might be more convenient to solve the corresponding Riccati *differential* equations when Eq. (73)—without the steady state marks on B_{ij}—is substituted into Eqs. (72a–c).)

Figure 4 displays a computed case with $y_2 = 0$, $R_{11} = 0$ and $R_{22} = Q_{11} = 1$, using the corresponding minimum steady state variance

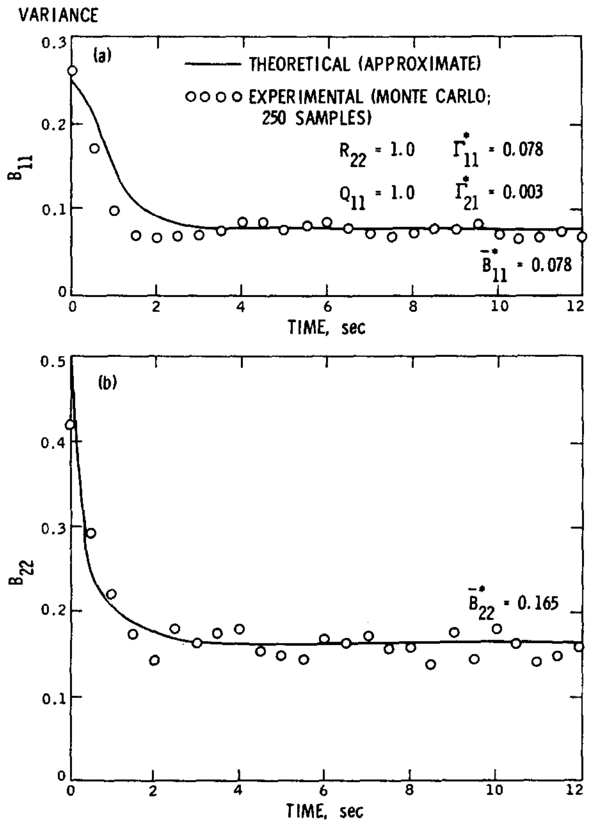

FIG. 4. Method I (vector problem).

constant filter gains Γ_{11}^* and Γ_{21}^*. The "theoretical" curves on Fig. 4 are the solutions of Eqs. (72a–c) with the relevant noise parameters and constant filter gains. For the sake of comparison, Fig. 4 also displays the results of the Monte Carlo simulation of the filter. Again, it is seen in Fig. 4 that the "experimental" values do indeed converge to the analytically predicted values of the filtering error variances and, the steady state "experimental" and analytical values agree very well.

Appendix A

To simplify notations and algebra, consider a scalar Markov process $x(t)$ with conditional probability density $p(t, x \mid t_0 x_0)$. Let the moments of the conditional probability density be denoted as (supressing, for convenience, the conditioning arguments in the subsequent equations)

$$m(t) \triangleq \int xp(t, x)\, dx \tag{A.1}$$

$$\beta_n(t) \triangleq \int [x - m(t)]^n\, p(t, x)\, dx. \tag{A.2}$$

Furthermore, let an associated moment function $\lambda_n(t)$ be defined as

$$\lambda_n(t) \triangleq \int [x - m(t)]^n\, L^+[p(t, x)]\, dx, \tag{A.3}$$

where the forward Kolmogorov operator L^+ is defined by Eq. (13) in the main text.

1. *Time evolution of the mean*

The increment of m in Δ time is, by definition,

$$\begin{aligned} \delta m(t) &\triangleq m(t + \Delta) - m(t) \\ &= \int xp(t + \Delta, x)\, dx - \int xp(t, x)\, dx \\ &= \int x[p(t, x) + \delta p(t)]\, dx - \int xp(t, x)\, dx \\ &= \int x\delta p(t)\, dx. \end{aligned} \tag{A.4}$$

Since the increment (= total time derivative) $\delta p(t)$ of the conditional probability density satisfies the forward Kolmogorov equation, we can write

$$\begin{aligned} \delta m(t) &= \int x L^{+}[p(t, x)]\, dx \Delta \\ &= \int L^{-}[x]\, p(t, x)\, dx \Delta, \end{aligned} \tag{A.5}$$

where the last step follows from the adjoint property of the L^{+} and L^{-} differential operators. Dividing by Δ and taking the limit as $\Delta \to 0$, we obtain

$$\dot{m} = \int L^{-}[x]\, p(t, x)\, dx. \qquad \text{Q,E.D.} \tag{A.6}$$

Note that Eq. (A.6) can also be written as

$$\dot{m} = \lambda_1(t) \tag{A.7}$$

since, from Eq. (A.3), we have

$$\begin{aligned} \lambda_1(t) &= \int [x - m(t)]\, L^{+}[p(t, x)]\, dx \\ &= \int x L^{+}[p(t, x)]\, dx - m(t) \int L^{+}[p(t, x)]\, dx \\ &= \int x L^{+}[p(t, x)]\, dx. \end{aligned} \tag{A.8}$$

2. *Time evolution of the variance*

The increment of β_n in Δ time is, by definition,

$$\begin{aligned} \delta\beta_n(t) &\triangleq \beta_n(t + \Delta) - \beta_n(t) \\ &= \int [x - m(t + \Delta)]^n\, p(t + \Delta, x)\, dx - \int [x - m(t)]^n\, p(t, x)\, dx \\ &= \int [\{x - m(t)\} - \delta m(t)]^n\, [p(t, x) + \delta p(t)]\, dx \\ &\quad - \int [x - m(t)]^n\, p(t, x)\, dx \\ &= \int [\{x - m(t)\}^n - n\{x - m(t)\}^{n-1}\, \delta m(t) \\ &\quad - \tfrac{1}{2} n(n-1)\, \{x - m(t)\}^{n-2}\, \delta^2 m(t) + \cdots +]\, [p(t, x) + \delta p(t)]\, dx \\ &\quad - \int [x - m(t)]^n\, p(t, x)\, dx. \end{aligned} \tag{A.9}$$

Now, for $n = 2$, Eq. (A.9) is reduced to

$$\begin{aligned}\delta\beta_2(t) = &\int [x - m(t)]^2\, \delta p(t)\, dx \\ &- 2\int [x - m(t)]\, \delta m(t)\, p(t, x)\, dx \\ &- 2\int [x - m(t)]\, \delta m(t)\, \delta p(t)\, dx \\ &- \int [p(t, x) + \delta p(t)]\, \delta^2 m(t)\, dx.\end{aligned} \tag{A.10}$$

Now, the second integral in Eq. (A.10) is obviously zero. The third integral can be written as

$$\int [x - m(t)]\, \lambda_1(t)\, \Delta L^+[p(t, x)]\, \Delta\, dx \sim 0(\Delta^2) \sim 0$$

and the fourth integral, using a similiar procedure as for the third integral, is $\sim 0(\Delta^3) \sim 0$. Hence, effective contribution comes only from the first integral in Eq. (A.10), where $\delta p(t)$ can be replaced by the Kolmogorov equations. Thus,

$$\begin{aligned}\delta\beta_2(t) &= \int [x - m(t)]^2\, L^+[p(t, x)\, dx\Delta \\ &= \int L^-[x - m(t)]^2\, p(t, x)\, dx\Delta.\end{aligned} \tag{A.11}$$

Dividing by Δ and taking the limit as $\Delta \to 0$ and, defining $b \triangleq \beta_2$, one obtains

$$\dot{b} = \int L^-[x - m(t)]^2\, p(t, x)\, dx. \qquad \text{Q.E.D.} \tag{A.12}$$

Appendix B

Applying Method II to the scalar filtering problem described by Eqs. (61–63), the differential equations for the components of the mean and variance of the filtering error (Eqs. (35–36) and (38a–c)) become, taking the case of nonlinear observations (Eq. (62b)),

$$\dot{\mu} = -\frac{\mu}{1 + \mu^2} + c_{11}\frac{\mu(3 - \mu^2)}{(1 + \mu^2)^3} \tag{B.1}$$

$$\dot{\nu} = -\frac{\nu}{1+\nu^2} + c_{22}\frac{\nu(3-\nu^2)}{(1+\nu^2)^3} + \gamma^*[\arctan(\mu) - \arctan(\nu)] + \gamma^*\left[\frac{\nu}{(1+\nu^2)^2}c_{22} - \frac{\mu}{(1+\mu^2)^2}c_{11}\right] \tag{B.2}$$

$$\dot{c}_{11} = 2c_{11}\frac{\mu^2-1}{(1+\mu^2)^2} + \sigma_s^2 \tag{B.3}$$

$$\dot{c}_{12} = c_{12}\left[\frac{\mu^2-1}{(1+\mu^2)^2} + \frac{\nu^2-1}{(1+\nu^2)^2}\right] + \gamma^*\left[\frac{c_{11}}{1+\mu^2} - \frac{c_{12}}{1+\nu^2}\right] \tag{B.4}$$

$$\dot{c}_{22} = 2c_{22}\frac{\nu^2-1}{(1+\nu^2)^2} + 2\gamma^*\left[\frac{c_{12}}{1+\mu^2} - \frac{c_{22}}{1+\nu^2}\right] + \gamma^{*2}\sigma_m^2, \tag{B.5}$$

where γ^* is the minimum steady state variance constant filter gain determined from Method I and given by Eq. (67). The initial conditions are $\nu_0 = \mu_0$, and $c_{22_0} = c_{11_0}$, $c_{12_0} = \frac{1}{2}c_{11_0}$, where μ_0 and c_{11_0} are given as part of the problem. Eqs. (B.1–5) should be compared to the *single* equation, Eq. (64), of Method I.

For the results depicted in Fig. 3 the initial conditions are $\mu_0 = 0.3$, $c_{11_0} = 0.64$.

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